

HILBERT COEFFICIENTS OF \mathfrak{m} -PRIMARY IDEALS IN ONE-DIMENSIONAL NOETHERIAN LOCAL RINGS

ASUKI KOURA AND NAOKI TANIGUCHI

ABSTRACT. This paper mainly studies the behavior of the first Hilbert coefficients $e_1(I)$ of \mathfrak{m} -primary ideals I in one-dimensional Noetherian local rings (A, \mathfrak{m}) . The purpose is to characterize those local rings A for which the first Hilbert coefficients $e_1(I)$ of \mathfrak{m} -primary ideals I sit within the range of finitely many values. Examples are explored.

1. INTRODUCTION

Let A be a commutative Noetherian local ring with maximal ideal \mathfrak{m} and $d = \dim A \geq 1$. For each \mathfrak{m} -primary ideal I in A and for each integer $n \geq 0$ we put

$$H_I(n) = \ell_A(A/I^{n+1}),$$

where $\ell_A(A/I^{n+1})$ denotes the length of A/I^{n+1} . Let us call $H_I(n)$ the Hilbert function of A with respect to I . Then, as is well-known, there exist integers $\{e_i(I)\}_{0 \leq i \leq d}$ such that

$$H_I(n) = e_0(I) \binom{n+d}{d} - e_1(I) \binom{n+d-1}{d-1} + \cdots + (-1)^d e_d(I)$$

for all $n \gg 0$. We call the integer $e_i(I)$ the i -th Hilbert coefficient of I . The positive integer $e_0(I) > 0$ is particularly called the multiplicity of A with respect to I and has been explored intensively. We put

$$\Lambda(A) = \{e_1(I) \mid I \text{ is an } \mathfrak{m}\text{-primary ideal in } A\}.$$

In this paper we study the behavior of the first Hilbert coefficients $e_1(I)$ of \mathfrak{m} -primary ideals I in one-dimensional Noetherian local rings and the main result of this paper is summarized into the following.

Theorem 1.1. *Let (A, \mathfrak{m}) be a Noetherian local ring with $d = \dim A \geq 1$. Then the following are equivalent.*

- (1) $\Lambda(A)$ is a finite set.
- (2) $d = 1$ and $A/H_{\mathfrak{m}}^0(A)$ is analytically unramified.

Here $H_{\mathfrak{m}}^0(A)$ denotes the 0-th local cohomology module of A with respect to \mathfrak{m} .

The classical theorem of M. Narita [3] shows that if A is a Cohen–Macaulay local ring, then $e_1(I) \geq 0$ for every \mathfrak{m} -primary ideal I in A . Therefore, if A is a Cohen–Macaulay local ring of dimension $d \geq 1$, the question of when the set $\Lambda(A)$ is finite is equivalent to asking when the supremum of the values $e_1(I)$ is finite. Our Theorem 1.1 settles the question, showing that such Cohen–Macaulay local rings are exactly of dimension one and analytically unramified.

We shall prove Theorem 1.1 in Section 3. Our key for the proof is the method of calculation of $e_1(I)$ in Cohen–Macaulay local rings of dimension one. The method is already known and not due to us (see, e.g., E. Matlis [2]). However, since it plays a very important role in our paper, in Section 2 we shall briefly explain the method, in a slightly different manner from that of [2].

In Section 2 we will also show the following, where \overline{A} denotes the integral closure of A in its total ring $Q(A)$ of fractions.

Proposition 1.2. *Let (A, \mathfrak{m}) be a Cohen–Macaulay local ring with $\dim A = 1$. Suppose that the residue class field A/\mathfrak{m} of A is infinite. Then the equality*

$\Lambda(A) = \{\ell_A(B/A) \mid A \subseteq B \subseteq \overline{A} \text{ is an intermediate ring which is a module-finite extension of } A\}$
holds true.

Here we only need the assumption in Proposition 1.2 that the residue class field is infinite, in order to make sure of the existence of a minimal reduction, for a given \mathfrak{m} -primary ideal, generated by a single element.

The heart of Theorem 1.1 is the following. Here we are able to delete the assumption that the residue class field is infinite, thanks to the standard technique of enlarging the residue class field, which we shall explain in Section 2.

Theorem 1.3. *Let (A, \mathfrak{m}) be a Cohen–Macaulay local ring with $\dim A = 1$. Then*

$$\sup \Lambda(A) = \ell_A(\overline{A}/A).$$

Therefore, $\Lambda(A)$ is a finite set if and only if A is analytically unramified.

By Theorem 1.3 we know the value $t = \sup \Lambda(A)$. However it is not necessarily true that

$$\Lambda(A) = \{n \in \mathbb{Z} \mid 0 \leq n \leq t\}.$$

Including such an example (Example 4.7), in Section 4 we will explore concrete examples, in order to illustrate our theorems.

2. PRELIMINARIES FOR THE PROOF OF THEOREM 1.1

Throughout this section, unless otherwise specified, let (A, \mathfrak{m}) be a Cohen–Macaulay local ring of dimension one. Let I be an \mathfrak{m} -primary ideal in A and assume that I contains a parameter ideal $Q = (a)$ as a reduction. Hence there exists an integer $r \geq 0$ such that $I^{r+1} = QI^r$. This assumption is automatically satisfied, if the residue class field A/\mathfrak{m} is infinite. We put

$$\frac{I^n}{a^n} = \left\{ \frac{x}{a^n} \mid x \in I^n \right\} \subseteq Q(A)$$

for $n \geq 0$ and let

$$B = A \left[\frac{I}{a} \right] \subseteq Q(A),$$

where $Q(A)$ denotes the total ring of fractions of A . Then, since $\frac{I^n}{a^n} \subseteq \frac{I^{n+1}}{a^{n+1}}$ for all $n \geq 0$, we get $\frac{I^n}{a^n} = \frac{I^r}{a^r}$ if $n \geq r$. Therefore

$$B = \bigcup_{n \geq 0} \frac{I^n}{a^n} = \frac{I^r}{a^r} \cong I^r$$

as an A -module. Hence the A -module B is finitely generated, so that $A \subseteq B \subseteq \overline{A}$.

The following Lemma 2.1 is the key for our proof. The result of the present form is due to [1, Lemma 2.1] (see also [2]). Let us note a brief proof for the sake of completeness.

Lemma 2.1 ([1, Lemma 2.1], [2]). *Let (A, \mathfrak{m}) be a Cohen-Macaulay local ring with $\dim A = 1$. Let I be an \mathfrak{m} -primary ideal in A and assume that I contains a parameter ideal $Q = (a)$ as a reduction. Choose an integer $r \geq 0$ such that $I^{r+1} = QI^r$. Then we have*

- (1) $e_0(I) = \ell_A(A/Q)$ and
- (2) $e_1(I) = \ell_A(I^r/Q^r) = \ell_A(B/A) \leq \ell_A(\overline{A}/A)$,

where $B = A\left[\frac{I}{a}\right]$.

Proof. Since

$$I^{n+1}/Q^{n+1} \cong \left[\frac{I^{n+1}}{a^{n+1}}\right]/A \subseteq B/A$$

for all $n \geq 0$, we get $I^r/Q^r \cong B/A$ and

$$\begin{aligned} \ell_A(A/I^{n+1}) &= \ell_A(A/Q^{n+1}) - \ell_A(I^{n+1}/Q^{n+1}) \\ &\geq \ell_A(A/Q^{n+1}) - \ell_A(I^r/Q^r) \\ &= \ell_A(A/Q^{n+1}) - \ell_A(B/A) \\ &= \ell_A(A/Q) \binom{n+1}{1} - \ell_A(B/A). \end{aligned}$$

Therefore

$$\ell_A(A/I^{n+1}) = \ell_A(A/Q) \binom{n+1}{1} - \ell_A(B/A),$$

if $n \geq r - 1$. □

Conversely, let $A \subseteq B \subseteq \overline{A}$ be an intermediate ring such that B is a finitely generated A -algebra. Choose a non-zero-divisor $a \in \mathfrak{m}$ of A so that $aB \subsetneq A$. We put $I = aB$. Then I is an \mathfrak{m} -primary ideal in A and

$$I^2 = (aB)^2 = a^2B = a(aB) = aI.$$

Since $B = A\left[\frac{I}{a}\right] = \frac{I}{a}$, by Lemma 2.1 we get the following.

Proposition 2.2.

$$\ell_A(B/A) = e_1(I) \in \Lambda(A).$$

We are now able to prove a part of Theorem 1.3, using Proposition 2.2.

Lemma 2.3. *Let (A, \mathfrak{m}) be a Cohen-Macaulay local ring with $\dim A = 1$. Then*

$$\sup \Lambda(A) \geq \ell_A(\overline{A}/A).$$

Proof. Let $t = \sup \Lambda(A)$ and assume that $t < \ell_A(\overline{A}/A)$. Then there exist $y_1, y_2, \dots, y_\ell \in \overline{A}$ such that $\ell_A(\left[\sum_{i=1}^\ell Ay_i\right]/A) > t$. Look at the ring $B = A[y_1, y_2, \dots, y_\ell]$. We then have $A \subseteq B \subseteq \overline{A}$ and

$$t < \ell_A\left(\left[\sum_{i=1}^\ell Ay_i\right]/A\right) \leq \ell_A(B/A),$$

which is impossible, since $\ell_A(B/A) \in \Lambda(A)$ by Proposition 2.2. Thus $t \geq \ell_A(B/A)$. \square

Let us prove Proposition 1.2.

Proof of Proposition 1.2. Let I be an \mathfrak{m} -primary ideal in A and choose a reduction $Q = (a)$ of I . We put $B = A\left[\frac{I}{a}\right]$. Then B is a module-finite extension of A and by Lemma 2.1 we get $e_1(I) = \ell_A(B/A)$. Hence

$$\Lambda(A) \subseteq \{\ell_A(B/A) \mid A \subseteq B \subseteq \overline{A} \text{ is an intermediate ring which is a module-finite extension of } A\}.$$

See Proposition 2.2 for the reverse inclusion. \square

We now explain the standard technique of enlarging the residue class field. In the rest of this section, let (A, \mathfrak{m}) be a Noetherian local ring. We consider the localization $A(X) = A[X]_{\mathfrak{p}}$ of the polynomial ring $A[X]$, where $\mathfrak{p} = \mathfrak{m}A[X]$. Then the composite map

$$A \longrightarrow A[X] \longrightarrow A(X) = A[X]_{\mathfrak{p}}$$

is a flat local homomorphism of Noetherian local rings. We put $\mathfrak{n} = \mathfrak{m}A(X)$, the maximal ideal of $A(X)$. Then, since $A(X)/\mathfrak{n} = (A/\mathfrak{m})(X)$, the residue class field $A(X)/\mathfrak{n}$ of $A(X)$ is infinite, $\dim A(X) = \dim A$, and for each A -module M

$$\ell_A(M) = \ell_{A(X)}(A(X) \otimes_A M).$$

Also, since $\mathfrak{n} = \mathfrak{m}A(X)$, $A(X)$ is a Cohen–Macaulay local ring if and only if A is a Cohen–Macaulay local ring. Therefore, if I is an \mathfrak{m} -primary ideal in A , then $IA(X)$ is an \mathfrak{n} -primary ideal in $A(X)$ and for all $n \geq 0$ we have

$$\begin{aligned} \ell_A(A/I^{n+1}) &= \ell_{A(X)}(A(X) \otimes_A (A/I^{n+1})) \\ &= \ell_{A(X)}(A(X)/(IA(X))^{n+1}). \end{aligned}$$

Hence, if $\dim A \geq 1$, we get

Fact 2.4.

$$e_0(I) = e_0(IA(X)) \quad \text{and} \quad e_1(I) = e_1(IA(X)).$$

Thus, when we consider Hilbert functions, we are able to assume, passing to the ring $A(X) = A[X]_{\mathfrak{m}A[X]}$, that the residue class field of A is infinite.

3. PROOF OF THEOREM 1.1

Let (A, \mathfrak{m}) be a Noetherian local ring with $\dim A = d \geq 1$. The purpose of this section is to prove Theorem 1.1. We begin with the partial proof of $(1) \Rightarrow (2)$ in Theorem 1.1, which asserts that if the set

$$\Lambda(A) = \{e_1(I) \mid I \text{ is an } \mathfrak{m}\text{-primary ideal of } A\}$$

is finite, then $d = 1$.

Partial proof of $(1) \Rightarrow (2)$ in Theorem 1.1. Let I be an \mathfrak{m} -primary ideal in A . Then for all integers $k \geq 1$ we get

$$e_0(I^k) = k^d \cdot e_0(I), \quad e_1(I^k) = \frac{d-1}{2} \cdot e_0(I) \cdot k^d + \frac{2e_1(I) - e_0(I) \cdot (d-1)}{2} \cdot k^{d-1}.$$

In fact, by the definition of Hilbert coefficient $e_i(I^k)$, we have

$$(1) \quad \ell_A(A/(I^k)^{n+1}) = e_0(I^k) \binom{n+d}{d} - e_1(I^k) \binom{n+d-1}{d-1} + \cdots + (-1)^d e_d(I^k)$$

for $n \gg 0$. On the other hand, we have

$$(2) \quad \begin{aligned} \ell_A(A/(I^k)^{n+1}) &= \ell_A(A/I^{(kn+k-1)+1}) \\ &= e_0(I) \binom{(kn+k-1)+d}{d} - e_1(I) \binom{(kn+k-1)+d-1}{d-1} \\ &\quad + \cdots + (-1)^d e_d(I), \end{aligned}$$

and

$$\begin{aligned} \binom{kn+k+d-1}{d} &= k^d \binom{n+d}{d} + \alpha \binom{n+d-1}{d-1} + (\text{lower terms}) \\ \binom{kn+k+d-2}{d-1} &= k^{d-1} \binom{n+d-1}{d-1} + (\text{lower terms}), \end{aligned}$$

where

$$\alpha = k^{d-1} \left(k + \frac{d-1}{2} \right) - \frac{k^d}{2} (d+1).$$

We now compare the coefficients of n^d in equations (1) and (2) and get

$$e_0(I^k) = k^d \cdot e_0(I).$$

Similarly, comparing those of n^{d-1} , we see

$$\begin{aligned} e_1(I^k) &= -e_0(I)\alpha + e_1(I)k^{d-1} \\ &= -e_0(I) \left(k^d + \frac{d-1}{2} k^{d-1} - \frac{d+1}{2} k^d \right) + e_1(I)k^{d-1} \\ &= \frac{d-1}{2} \cdot e_0(I) \cdot k^d + \frac{2e_1(I) - e_0(I) \cdot (d-1)}{2} \cdot k^{d-1}. \end{aligned}$$

Hence $d = 1$, if the set $\{e_1(I^k) \mid k \geq 1\}$ is finite. □

We are now in a position to prove Theorem 1.3.

Proof of Theorem 1.3. Thanks to Lemma 2.3, it is enough to show that $\sup \Lambda(A) \leq \ell_A(\overline{A}/A)$. We may assume $\ell_A(\overline{A}/A) < \infty$. Hence \overline{A} is a finitely generated A -module, so that \widehat{A} is reduced. Therefore $\overline{A[X]} = \overline{A}[X]$, where X is an indeterminate over A . Consequently

$$(\overline{A[X]})_{\mathfrak{p}} = \overline{A[X]_{\mathfrak{p}}} = \overline{A(X)},$$

where $\mathfrak{p} = \mathfrak{m}A[X]$. Let $B = A(X)$ and $\mathfrak{n} = \mathfrak{m}B$. Hence (B, \mathfrak{n}) is a Cohen–Macaulay local ring with $\dim B = 1$. Let I be an \mathfrak{m} -primary ideal in A . Then, since the residue class field B/\mathfrak{n} is infinite, there exists $a \in IB$ such that aB is a reduction of IB . Therefore

$$e_1(I) = e_1(IB) \leq \ell_B(\overline{B}/B)$$

by Lemma 2.1(2) and Fact 2.4. On the other hand, because $(\overline{A[X]})_{\mathfrak{p}} = \overline{B}$ and B is a flat A -module, we obtain the following exact sequence

$$\begin{array}{ccccccc} 0 & \rightarrow & B \otimes_A A & \rightarrow & B \otimes_A \overline{A} & \rightarrow & B \otimes_A \overline{A}/A \rightarrow 0 \\ & & \parallel & & \parallel & & \parallel \\ & & B & & \overline{B} & & \overline{B}/B \end{array}$$

of B -modules. Therefore we have

$$e_1(I) = e_1(IB) \leq \ell_B(\overline{B}/B) = \ell_A(\overline{A}/A)$$

and hence

$$\sup \Lambda(A) = \ell_A(\overline{A}/A)$$

as claimed.

Since the analytical unramifiedness of A is equivalent to the finite generation of the A -module \overline{A} , $\Lambda(A)$ is finite if and only if \widehat{A} is a reduced ring. This completes the proof of Theorem 1.3. \square

When the base ring A is not necessarily Cohen–Macaulay, Theorem 1.3 is stated as follows.

Theorem 3.1. *Let (A, \mathfrak{m}) be a Noetherian local ring with $\dim A = 1$. Then*

$$\begin{aligned} \sup \Lambda(A) &= \ell_B(\overline{B}/B) - \ell_A(H_{\mathfrak{m}}^0(A)) \quad \text{and} \\ \inf \Lambda(A) &= -\ell_A(H_{\mathfrak{m}}^0(A)), \end{aligned}$$

where $B = A/H_{\mathfrak{m}}^0(A)$.

Proof. Let $W = H_{\mathfrak{m}}^0(A)$. Then $B = A/W$ is a one-dimensional Cohen–Macaulay local ring. Let I be an \mathfrak{m} -primary ideal in A and look at the exact sequence

$$0 \rightarrow W/[I^{n+1} \cap W] \rightarrow A/I^{n+1} \rightarrow B/I^{n+1}B \rightarrow 0$$

of A -modules. Then since $I^{n+1} \cap W = (0)$ for all $n \gg 0$, we get

$$\begin{aligned} \ell_A(A/I^{n+1}) &= \ell_A(B/I^{n+1}B) + \ell_A(W) \\ &= e_0(IB) \binom{n+1}{1} - e_1(IB) + \ell_A(W). \end{aligned}$$

Hence

$$e_0(I) = e_0(IB) \quad \text{and} \quad e_1(I) = e_1(IB) - \ell_A(W) \geq -\ell_A(W),$$

since $e_1(IB) \geq 0$ by [3]. If I is a parameter ideal in A , then IB is a parameter ideal in B and we get

$$e_1(I) = e_1(IB) - \ell_A(W) = -\ell_A(W).$$

Therefore, since every $\mathfrak{m}B$ -primary ideal J in B has the form $J = IB$ for some \mathfrak{m} -primary ideal I in A , we readily get the required estimations

$$\begin{aligned} \sup \Lambda(A) &= \sup \Lambda(B) - \ell_A(W) \quad \text{and} \\ &= \ell_B(\overline{B}/B) - \ell_A(W) \\ \inf \Lambda(A) &= -\ell_A(W). \end{aligned}$$

□

The following is a direct consequence of Theorem 3.1.

Corollary 3.2. *Let (A, \mathfrak{m}) be a one-dimensional Noetherian local ring. Then $\Lambda(A)$ is finite if and only if $B = A/H_{\mathfrak{m}}^0(A)$ is analytically unramified.*

Let us finish the proof of Theorem 1.1.

Proof of Theorem 1.1. The implication $(2) \Rightarrow (1)$ in Theorem 1.1 now follows from Corollary 3.2, which completes the proof of Theorem 1.1, because $d = 1$, thanks to the partial proof of $(1) \Rightarrow (2)$ in Theorem 1.1 above. □

4. EXAMPLES

Let us explore examples.

Let a_1, a_2, \dots, a_ℓ ($\ell \geq 1$) be positive integers such that $\text{GCD}(a_1, a_2, \dots, a_\ell) = 1$ and let $V = k[[t]]$ be the formal power series ring over a field k . We put $A = k[[t^{a_1}, t^{a_2}, \dots, t^{a_\ell}]]$ and $H = \langle \sum_{i=1}^\ell c_i a_i \mid 0 \leq c_i \in \mathbb{Z} \rangle$. Hence A is the semigroup ring of the numerical semigroup H . We have $V = \overline{A}$ and $\ell_A(V/A) = \sharp(\mathbb{N} \setminus H)$, where $\mathbb{N} = \{n \mid 0 \leq n \in \mathbb{Z}\}$. Let $c = c(H)$ be the conductor of H .

Example 4.1. Let $q = \sharp(\mathbb{N} \setminus H)$. Then

$$\Lambda(A) = \{0, 1, \dots, q\}.$$

Proof. We may assume $q \geq 1$. Hence $c \geq 2$. Let us write $S = \{c_1, c_2, \dots, c_q\}$ with $1 = c_1 < c_2 < \dots < c_q = c - 1$ and put $B_i = A[t^{c_i}, t^{c_{i+1}}, \dots, t^{c_q}]$ for each $1 \leq i \leq q$. We then have the descending chain

$$V = B_1 \supsetneq B_2 \supsetneq \dots \supsetneq B_q \supsetneq B_{q+1} := A$$

of A -algebras, which gives rise to a composition series $\{B_i/A\}_{1 \leq i \leq q+1}$ of the A -module V/A , since $\ell_A(V/A) = q$. Hence $\ell_A(B_i/A) = q + 1 - i$ for all $1 \leq i \leq q + 1$ and therefore, setting $a = t^c$ and $I_i = aB_i$ ($\subsetneq A$), by Proposition 2.2 we get $e_1(I_i) = q + 1 - i$. Thus $\Lambda(A) = \{0, 1, \dots, q\}$ as claimed. □

Because $q = \frac{c(H)}{2}$ if H is symmetric (that is $A = k[[t^{a_1}, t^{a_2}, \dots, t^{a_\ell}]]$ is a Gorenstein ring), we readily have the following.

Corollary 4.2. *Suppose that H is symmetric. Then $\Lambda(A) = \{0, 1, \dots, \frac{c(H)}{2}\}$.*

We shall explore one more numerical semigroup ring.

Corollary 4.3. *Let $A = k[[t^n, t^{n+1}, \dots, t^{2n-1}]]$ ($n \geq 2$). Then $\Lambda(A) = \{0, 1, \dots, n-1\}$. For the ideal $I = (t^n, t^{n+1}, \dots, t^{2n-2})$ of A , one has*

$$e_1(I) = \begin{cases} r(A) - 1 & (n = 2), \\ r(A) & (n \geq 3). \end{cases}$$

Proof. See Example 4.1 for the first assertion. Let us check the second one. If $n = 2$, then the ring A is Gorenstein and I is a parameter ideal in A . Hence $e_1(I) = r(A) - 1 (= 0)$. Let $n \geq 3$ and put $Q = (t^n)$. Then Q is a reduction of I , since $IV = QV$. Because

$$A\left[\frac{I}{a}\right] = k[[t]]$$

and $\mathfrak{m} = t^n V$, we get $A :_{Q(A)} \mathfrak{m} = k[[t]]$. Hence

$$e_1(I) = \ell_A(k[[t]]/A) = \ell_A([A :_{Q(A)} \mathfrak{m}]/A) = r(A),$$

as claimed. \square

Remark 4.4. In Example 4.3 I is a canonical ideal of A . Therefore the equality $e_1(I) = r(A)$ shows that if $n \geq 3$, A is not a Gorenstein ring but an almost Gorenstein ring in the sense of [1] (see [1, Corollary 3.12] also).

We now consider the case where the base rings are not analytically irreducible.

Example 4.5. Let (R, \mathfrak{n}) be a regular local ring with $n = \dim R \geq 2$. Let X_1, X_2, \dots, X_n be a regular system of parameters in S and put for each $1 \leq i \leq n$, $P_i = (X_j \mid 1 \leq j \leq n, j \neq i)$. We look at the ring

$$A = R \Big/ \bigcap_{i=1}^n P_i.$$

Then A is a one-dimensional Cohen–Macaulay local ring with $\Lambda(A) = \{0, 1, \dots, n-1\}$.

Proof. Let x_i denote the image of X_i in A . We put $\mathfrak{p}_i = (x_j \mid 1 \leq j \leq n, j \neq i)$ and $B = \prod_{i=1}^n (A/\mathfrak{p}_i)$. Then the homomorphism

$$\varphi : A \rightarrow B, \quad a \mapsto (\bar{a}, \bar{a}, \dots, \bar{a})$$

is injective and $B = \overline{A}$. Since $\mathfrak{m}B = \mathfrak{m}$ and $\mu_A(B) = n$, we get $\ell_A(B/A) = n - 1$. Let $\mathbf{e}_j = (0, \dots, 0, \overset{j}{1}, 0, \dots, 0)$ for $1 \leq j \leq n$ and $\mathbf{e} = \sum_{j=1}^n \mathbf{e}_j$. Then $B = A\mathbf{e} + \sum_{j=1}^{n-1} A\mathbf{e}_j$. We put

$$B_i = A\mathbf{e} + \sum_{j=1}^i A\mathbf{e}_j$$

for each $1 \leq i \leq n - 1$. Then since $B_i = A[\mathbf{e}_1, \mathbf{e}_2, \dots, \mathbf{e}_i]$, B_i is a finitely generated A -algebra and $B_i \subsetneq B_{i+1}$. Therefore the series

$$B = B_{n-1} \supsetneq B_{n-2} \supsetneq \dots \supsetneq B_1 \supsetneq B_0 := A$$

gives rise to a composition series of the A -module B/A . Hence $\ell_A(B_i/A) = i$ for all $0 \leq i \leq n - 1$ and thus $\Lambda(A) = \{0, 1, \dots, n - 1\}$. \square

Let A be an one-dimensional Cohen–Macaulay local ring. If A is not a reduced ring, then the set $\Lambda(A)$ must be infinite. Let us explore one concrete example with $\Lambda(A)$ infinite.

Example 4.6. Let V be a discrete valuation ring and let $A = V \ltimes V$ denote the idealization. Then $\Lambda(A) = \{n \mid 0 \leq n \in \mathbb{Z}\}$.

Proof. Let $K = Q(V)$. Then $Q(A) = K \ltimes K$ and $\overline{A} = V \ltimes K$. We put $B_n = V \ltimes \left(V \cdot \frac{1}{t^n}\right)$ for $n \geq 0$. Then $A \subseteq B_n \subseteq \overline{A}$. We have

$$\begin{aligned} \ell_A(B_n/A) &= \ell_V(B_n/A) \\ &= \ell_V([V \oplus \left(V \cdot \frac{1}{t^n}\right)]/[V \oplus V]) \\ &= \ell_V(V \cdot \frac{1}{t^n}/V) \\ &= \ell_V(V/t^n V) \\ &= n. \end{aligned}$$

Hence, thanks to Proposition 2.2, we get $n \in \Lambda(A)$ and therefore

$$\Lambda(A) = \{n \mid 0 \leq n \in \mathbb{Z}\}.$$

□

We close this paper with the following.

Example 4.7. Let K/k ($K \neq k$) be a finite extension of fields and assume that there are no proper intermediate fields between K and k . Let $n = [K : k]$ and choose a k -basis $\{\omega_i\}_{1 \leq i \leq n}$ of K . Let $K[[t]]$ be the formal power series ring over K and put $A = k[[\omega_1 t, \omega_2 t, \dots, \omega_n t]] \subseteq K[[t]]$. Then $\sharp \Lambda(A) = 2$. We actually have

$$\Lambda(A) = \{0, n-1\}.$$

Proof. Let $V = k[[t]]$. Then $V = \sum_{i=1}^n A\omega_i$ and $V = \overline{A}$. Since $tV \subseteq A$, we see $\mathfrak{n} = tV = \mathfrak{m}$, where \mathfrak{m} and \mathfrak{n} respectively denote the maximal ideals in A and V . Hence $\ell_A(V/A) = n-1$. Let $A \subseteq B \subseteq V$ be an intermediate ring. Then B is a local ring, whose maximal ideal is denoted by \mathfrak{m}_B . We then have $\mathfrak{m} = \mathfrak{m}_B = \mathfrak{n}$, since $\mathfrak{m} = \mathfrak{n}$. Therefore, looking at the extension of residue class fields

$$k = A/\mathfrak{n} \subseteq B/\mathfrak{n} \subseteq K = V/\mathfrak{n},$$

we readily get $V = B$ or $B = A$. Since $V = \overline{A}$ is a discrete valuation ring, every \mathfrak{m} -primary ideal in A contains a reduction generated by a single element. Hence

$$\Lambda(A) = \{0, n-1\}$$

by Proposition 1.2. □

REFERENCES

- [1] S. Goto, N. Matsuoka and T. T. Phuong, *Almost Gorenstein rings*, Preprint 2011.
- [2] E. Matlis, *One-dimensional Cohen-Macaulay Rings*, Lecture Notes in Mathematics, **327**, Springer-Verlag, (1973).
- [3] M. Narita, *A note on the coefficients of Hilbert characteristic functions in semi-regular local rings*, *Proc. Cambridge Philos. Soc.*, **59**(1963), 269–275.

OTSUMA NAKANO JUNIOR AND SENIOR HIGH SCHOOL, 2-3-7, KAMITAKATA, NAKANO-KU, TOKYO 164-0002, JAPAN

E-mail address: koura@otsumanakano.ac.jp

DEPARTMENT OF MATHEMATICS, GRADUATE COURSES, SCHOOL OF SCIENCE AND TECHNOLOGY, MEIJI UNIVERSITY, 1-1-1 HIGASHI-MITA, TAMA-KU, KAWASAKI 214-8571, JAPAN

E-mail address: taniguti@math.meiji.ac.jp